Differentiating Eq. (6) with regard to t, taking the divergence of Eq. (11), and subtracting the result from the first, one gets

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i \partial x_j}(\rho q_i q_i + P_{ij}) + \frac{\partial}{\partial t}[\rho_0 v_n |\nabla f| \delta(f)] - \frac{\partial}{\partial x_i}[p_{ij} n_j |\nabla f| \delta(f)]$$

Next, adding and subtracting the term $c_0^2 \nabla^2 \rho$ where c_0 is the velocity of sound in the undisturbed medium and ∇^2 is the Laplacian, one may write the above equation as

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \nabla^2 \rho = \frac{\partial^2}{\partial x_i \partial x_j}(\rho q_i q_j + P_{ij} - c_0^2 \rho \delta_{ij}) + \frac{\partial}{\partial t}[\rho_0 v_n |\nabla f| \delta(f)] - \frac{\partial}{\partial x_i}[p_{ij} n_j |\nabla f| \delta(f)]$$

Finally, as ρ_0 and p_0 are constants, their time and space derivatives are zero. Therefore, writing ρ as $\tilde{\rho}$ on the left-hand side of the above equation, where $\tilde{\rho}$ is the perturbation density defined by $\tilde{\rho} = \rho - \rho_0$, and writing $P_{ij} - c_0^2 \rho \delta_{ij}$ in the first term on the right as $p_{ij} - c_0^2 \tilde{\rho} \delta_{ij}$, one obtains the Ffowcs Williams and Hawkings equation

$$\frac{\partial^2 \tilde{\rho}}{\partial t^2} - c_0^2 \nabla^2 \tilde{\rho} = \frac{\partial^2}{\partial x_i \partial x_j} T_{ij} + \frac{\partial}{\partial t}[\rho_0 v_n |\nabla f| \delta(f)] - \frac{\partial}{\partial x_i}[p_{ij} n_j |\nabla f| \delta(f)] \tag{12}$$

where T_{ij} is the Lighthill stress tensor, defined by

$$T_{ij} = \rho q_i q_j + p_{ij} - c_0^2 \tilde{\rho} \delta_{ij} \tag{13}$$

Acknowledgments

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Transverse Vibrations of Nonuniform Rectangular Orthotropic Plates

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Introduction

INCREASING use is being made of nonisotropic and nonuniform elastic plates in the design of modern missiles, space vehicles, aircraft wings, and numerous composite engineering structures. The investigation presented here gives extensive and accurate results to study the transverse vibrations of a rectangular orthotropic plate of parabolically varying thickness. The governing differential equation of motion is obtained and solved by the Frobenius method to find the first three modes of vibration of a nonuniform rectangular orthotropic plate having different combinations of boundary conditions and for the various values of the taper parameter and length-to-breadth ratio. Some related work is listed in Refs. 1-3.

Equation of Motion

The following differential equation of motion for a nonuniform orthotropic plate is obtained,

$$D_x \frac{\partial^4 w}{\partial x^4} + D_y \frac{\partial^4 w}{\partial y^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2\frac{\partial H}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2} + 2\frac{\partial H}{\partial y}\frac{\partial^3 w}{\partial y \partial x^2} + 2\frac{\partial D_x}{\partial x}\frac{\partial^3 w}{\partial x^3} + 2\frac{\partial D_y}{\partial y}\frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 D_x}{\partial x^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 D_y}{\partial y^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 D_1}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 D_1}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + 4\frac{\partial^2 D_{xy}}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{1}$$

where

$$D_x = E_1 \frac{h^3}{12}, \quad D_y = E_2 \frac{h^3}{12}, \quad D_1 = E_3 \frac{h^3}{12}, \quad D_{xy} = G_{xy}\frac{h^3}{12}$$

$$H = D_1 + 2D_{xy}$$

Are the rigidity parameters in the appropriate directions of the orthotropy. For convenience we write here,

$$E_1 = \frac{E_x}{(1-\gamma_{xy}\gamma_{yx})}, \qquad E_2 = \frac{E_y}{(1-\gamma_{yx}\gamma_{xy})}$$

$$E_3 = \gamma_{xy} D_y = \gamma_{yx} D_x$$

Further, w is the transverse deflection of the plate, ρ the mass density per unit volume, h the plate thickness, and E_x, E_y and γ_x, γ_y are, respectively, the Young's modulii and Poisson's

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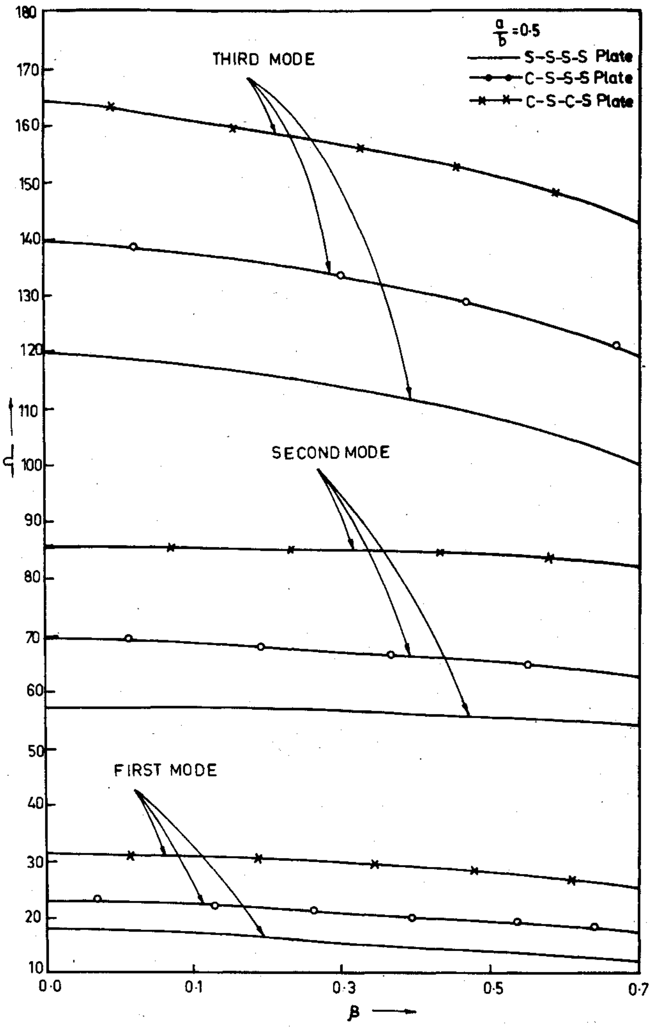

Fig. 1 Variation of frequency with taper parameter for the first three modes of vibration.

ratios in the two directions of the orthotropy of the plate material.

Let the two opposite edges $y=0$ and $y=b$ of the plate of length a and breadth b be simply supported. Then for harmonic vibrations,

$$w(x,y,t) = \bar{W}(x)\sin(m\pi y/b)e^{ipt} \tag{2}$$

is substituted in Eq. (1), where p is the circular frequency of vibration and m a positive integer. Then introduce

$$X=x/a, \quad Y=y/b, \quad \bar{h}=h/a \quad \text{and} \quad W=\bar{W}/a \tag{3}$$

as nondimensional variables and take parabolically varying thickness in the X direction as

$$\bar{h}=h_0(1-\beta X^2) \tag{4}$$

where $h_0=h|_{x=0}$ and β is the taper parameter.

Substitution of Eqs. (2-4) into Eq. (1) gives

$$(1-\beta X^2)^2\frac{d^4W}{dX^4}-12\beta X(1-\beta X^2)\frac{d^3W}{dX^3}$$
$$+[(30\beta^2X^2-6\beta)-2T_1^{(1)}\lambda^2(1-\beta X^2)^2]\frac{d^2W}{dX^2}$$
$$+12T_1^{(1)}\lambda^2\beta X(1-\beta X^2)\frac{dW}{dX}+[T_1^{(2)}\lambda^4(1-\beta X^2)^2$$
$$-3T_1^{(3)}\lambda^2(10\beta^2X^2-2\beta)-\Omega^2]W=0 \tag{5}$$

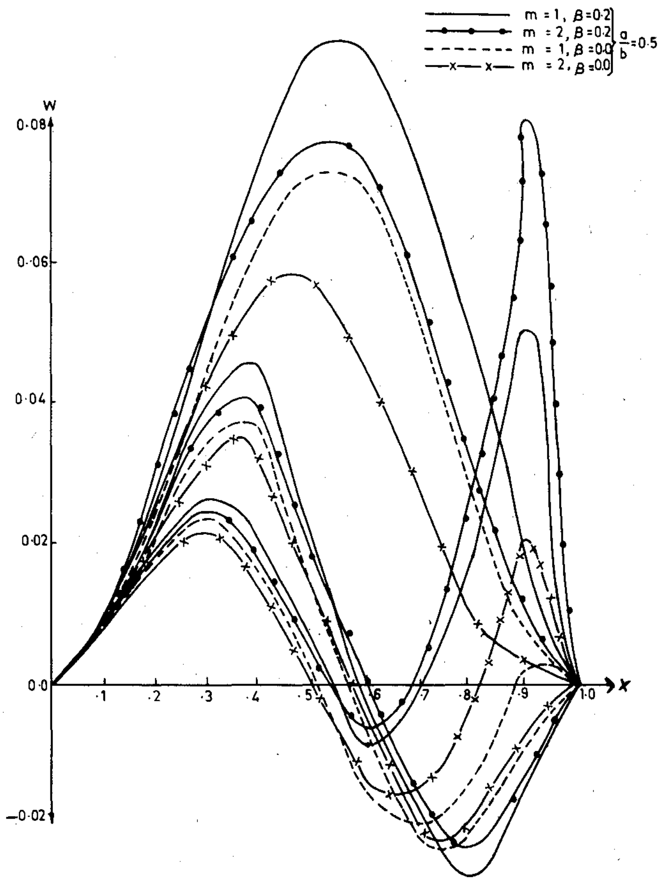

Fig. 2 Transverse deflection of C-S-C-S plates in the first three modes of vibration.

where

$$T_1^{(1)}=\frac{E_3+2G}{E_1}, \quad T_1^{(2)}=\frac{E_2}{E_1}, \quad T_1^{(3)}=\frac{E_3}{E_1}, \quad \lambda^2=\frac{m^2\pi^2a^2}{b^2}$$

and $\Omega^2=12\rho a^2p^2/E_1h_0^2$ is the nondimensional frequency parameter.

Method of Solution

The differential equation of motion (5) is solved by Forbenius method by assuming that

$$W=\sum_{r=0}^{\infty}a_rX^{k+r}, \qquad a_0\neq 0 \tag{6}$$

Substituting Eq. (6) into Eq. (5) and using the method of Frobenius, unknown constants a_r $(r=0,1,2,...)$ are determined.

It is seen that even coefficients involve a_0 and a_2, while the odd coefficients involve a_1 and a_3. Therefore,

$$a_{2i+2}=f_ia_0+\phi_ia_2 \quad \text{and} \quad a_{2i+3}=g_ia_1+\Psi_ia_3, \quad i=0,1,2,3...$$

Here f_i, g_i, ϕ_i, and Ψ_i are the functions of β,Ω,K, etc. It is noted that $f_0=g_0=0$ and $\phi_0=\Psi_0=1$.

Finally the solution is obtained as

$$W=a_0P_0(X,\Omega)+a_1P_1(X,\Omega)+a_2P_2(X,\Omega)+a_3P_3(X,\Omega) \tag{7}$$

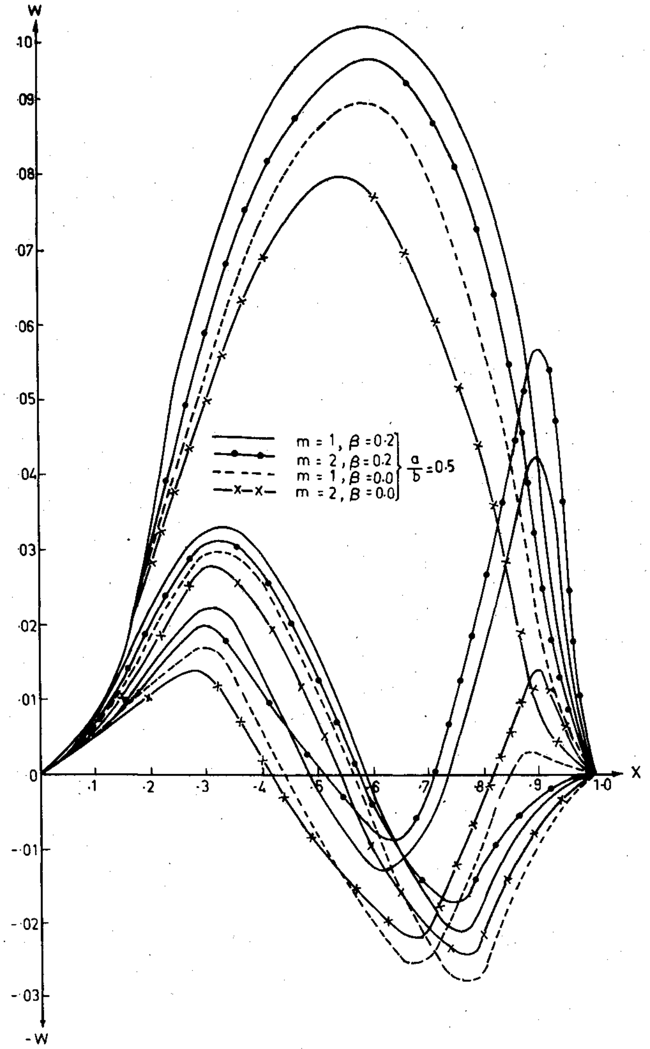

Fig. 3 Transverse deflection of C-S-S-S plates in the first three modes of vibration.

where

$$P_0(X,\Omega) = 1 + \sum_{i=1}^{\infty} f_i X^{2i+2}$$

$$P_1(X,\Omega) = X + \sum_{i=1}^{\infty} g_i X^{2i+3}$$

$$P_2(X,\Omega) = \sum_{i=0}^{\infty} \phi_i X^{2i+2}$$

$$P_3(X,\Omega) = \sum_{i=0}^{\infty} \Psi_i X^{2i+3}$$

Application of the technique used by Lamb[4] shows that Eq. (7) is convergent for all $|\beta| < 1$.

Boundary Conditions, Frequency Equations, and Deflections

Applying the appropriate boundary conditions to Eq. (7), one obtains the following equations for frequency and deflection, respectively, keeping the edges $y=0$ and $y=b$ simply supported.

1) C-S-C-S plate: Plate clamped at both the edges, $X=0$ and $X=1$,

$$P_2(1,\Omega)P_3'(1,\Omega) - P_3(1,\Omega)P_2'(1,\Omega) = 0 \quad (8)$$

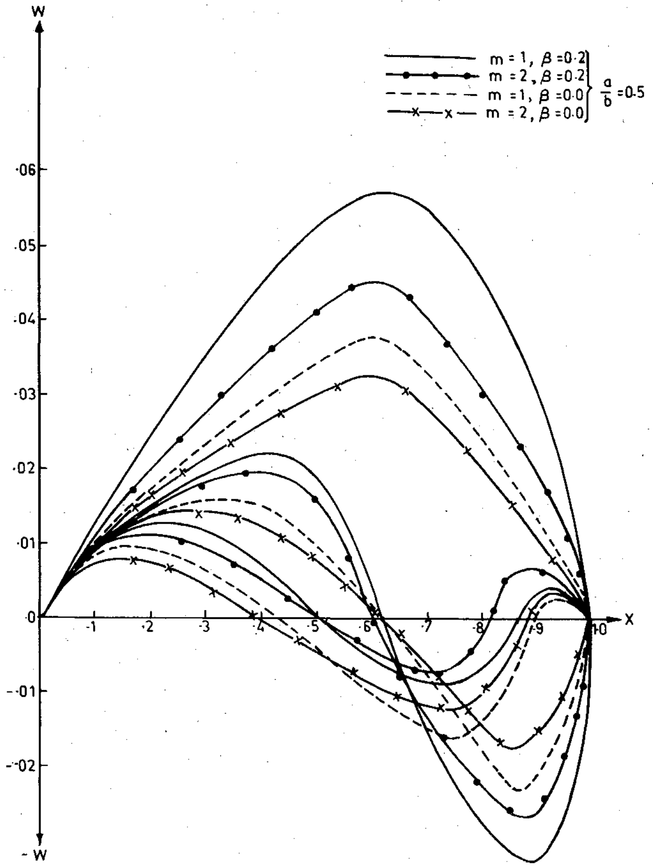

Fig. 4 Transverse deflection of S-S-S-S plates in the first three modes of vibration.

and

$$W = P_2(X,\Omega) - \frac{P_2(1,\Omega)}{P_3(1,\Omega)} P_3(X,\Omega) \quad (9)$$

2) C-S-S-S plate: Plate clamped at $X=0$ and simply supported at $X=1$,

$$P_2(1,\Omega)P_3''(1,\Omega) - P_3(1,\Omega)P_2''(1,\Omega) = 0 \quad (10)$$

and

$$W = P_2(X,\Omega) - \frac{P_2(1,\Omega)}{P_3(1,\Omega)} P_3(X,\Omega) \quad (11)$$

3) S-S-S-S plate: Plate simply supported at both edges, $X=0$ and $X=1$,

$$P_1(1,\Omega)P_3''(1,\Omega) - P_3(1,\Omega)P_1''(1,\Omega) = 0 \quad (12)$$

and

$$W = P_1(X,\Omega) - \frac{P_1(1,\Omega)}{P_3(1,\Omega)} P_3(X,\Omega) \quad (13)$$

Here primes denote the differentiation with respect to X.

Results and Discussion

The effect of parabolic nonuniformity has been studied on the frequency parameter ($\Omega = \sqrt{12\rho a^2 p^2 / E_1 h_0^2}$) and the transverse deflections of a rectangular orthotropic plate for the first three modes of vibration with different combinations of boundary conditions for various values of taper parameter β and length-to-breadth ratios a/b, taking integer values of

$m=1$ and 2. The calculations are based upon the properties of an orthotropic material, namely 5 ply maple plywood. Frequency curves (frequency vs taper parameter) and transverse deflection curves for the first three modes of vibration with different boundary conditions are plotted in Figs. 1-4.

It is interesting to note that Ω increases with the increase in a/b, whereas Ω decreases with the increase in β. This is found true for all three modes of vibration and for all three edge conditions. Also Ω corresponding to β decreases slowly in the first and second modes but in the third mode it decreases rapidly. It is also found that Ω for the C-S-C-S plates is greater than the corresponding Ω for the C-S-S-S and S-S-S-S plates. This difference increases with the move toward the higher modes of vibration. For comparison purposes, Ω has been computed for an isotropic plate of parabolically varying thickness by converting the orthotropic parameters into the usual isotropic parameters. It compares well with the Ω of Ref. 1 under the identical conditions.

It is evident from Figs. 2-4 that the deflection of a plate of variable thickness is more than that for a plate of uniform thickness. This difference between the deflections increases with the increase in β. The nodal lines have also shifted toward the thinner side of the plate.

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Stability of Short Beck and Leipholz Columns on Elastic Foundation

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Introduction

STABILITY of nonconservative systems is extensively discussed in Ref. 1. It has been shown[2,3] that for slender uniform columns, resting on an elastic foundation of constant foundation modulus and subjected to nonconservative loads, the critical loads remain the same regardless of the value of the foundation modulus, and that the coalescence frequency† shifts by a quantity equal to the foundation modulus, compared to the column with no elastic foundation. The purpose of the present Note is to study the effect of shear deformation and rotatory inertia on a Beck column (a cantilever column with a concentrated follower force at the free end) and a Leipholz column (a cantilever column with a uniformly distributed follower force) on an elastic foundation and to examine whether the aforementioned phenomenon is applicable for short columns. The finite-element formulation presented in this Note for including the effects of shear deformation and rotatory inertia follows the same lines as Ref. 4 and the nonconservative stability problem is formulated using the standard formulation of Refs. 5 and 6.

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†The load on the column at which the two lowest frequencies (in the present study) become complex is the critical load and the corresponding frequency is the coalescence frequency.

Finite-Element Formulation

The matrix equation governing the present nonconservative stability problem is obtained as[5]

$$(\lambda^2+\Omega)[M]\{q\}-[K]\{q\}+Q([G^C]+[G^{NC}])\{q\}=0 \quad (1)$$

where $[K]$, $[M]$, $[G^C]$, $[G^{NC}]$, and $\{q\}$ are the assembled elastic stiffness matrix, mass matrix, geometric stiffness matrix for the conservative part of the load, geometric stiffness matrix for the nonconservative part of the load, and eigenvector, respectively. In Eq. (1), $\lambda^2=m\omega^2L^4/EI$, where m is the mass per unit length, ω the circular frequency, L the length of the column, E the Young's modulus, and I the moment of inertia, and $\Omega=kL^4/EI$, where k is the foundation modulus per unit length. For Beck's column, $Q=PL^2/\pi^2EI$ and for Leipholz's column, $Q=pL^3/\pi^2EI$, where P is the concentrated tip load and p the distributed load on the column per unit length.

The element stiffness matrix $[k]$, the mass matrix $[m]$, and the geometric stiffness matrices $[g^C]$ and $[g^{NC}]$ are obtained by using the standard procedure[7] from the expressions

$$U=\frac{1}{2}\int_0^l\left[EI\psi_x^2+5/6\left(\frac{EA}{2(1+\nu)}\epsilon_{xz}^2\right)\right]\mathrm{d}x \quad (2)$$

Table 1 Critical loads Q_{cr} and coalescence frequencies λ^2_{cr} for Beck's column for various L/r and Ω, eight-element solution

L/r	Ω	Q_{cr}	λ^2_{cr}
15	0.0	1.40	71.7
	0.1	1.40	71.8
	1.0	1.40	72.7
	10.0	1.39	81.6
	100.0	1.35	170.4
	1000.0	0.901	1048
25	0.0	1.74	97.6
	0.1	1.74	97.7
	1.0	1.74	98.6
	10.0	1.74	107.6
	100.0	1.72	196.9
	1000.0	1.51	1090
50	0.0	1.95	114.4
	0.1	1.95	114.5
	1.0	1.95	115.4
	10.0	1.95	124.4
	100.0	1.94	214.3
	1000.0	1.88	1112
100	0.0	2.01	119.5
	0.1	2.01	119.6
	1.0	2.01	120.5
	10.0	2.01	129.5
	100.0	2.01	219.5
	1000.0	1.99	1119
500	0.0	2.03	121.1
	0.1	2.03	121.2
	1.0	2.03	122.1
	10.0	2.03	131.1
	100.0	2.03	221.1
	1000.0	2.03	1121